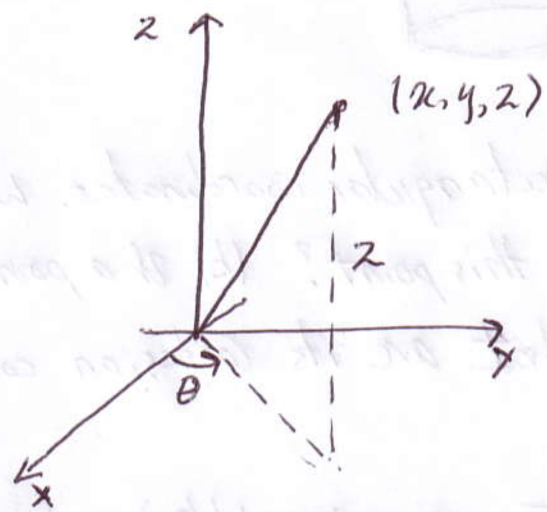
Cylindrical Coordinates

Def: The cylindrical coordinates (r, θ, z) of a point (x, y, z) are defined by

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

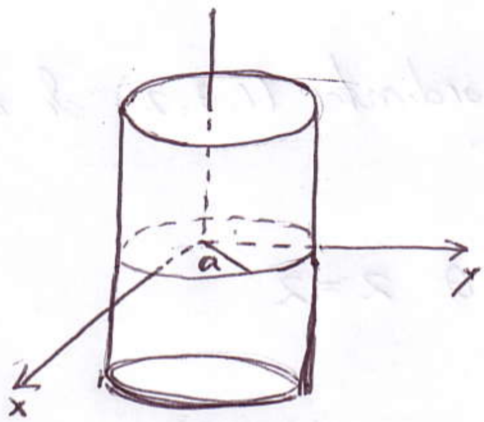


To express r, θ, and z in terms of x, y, z and to ensure that θ lies between 0 and 2π, we can write

$$r = \sqrt{x^2+y^2}, \quad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x>0 \text{ \& } y \geq 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x<0 \\ 2\pi + \tan^{-1}(y/x) & \text{if } x>0 \text{ \& } y<0 \end{cases}$$

The cylindrical coordinates are a natural extension to polar coordinates. They are called cylindrical, because the easiest surfaces described by means of these coordinates are cylinders.

For example, the graph of the points whose cylindrical coordinates satisfy the equation $r=a$ is a cylinder

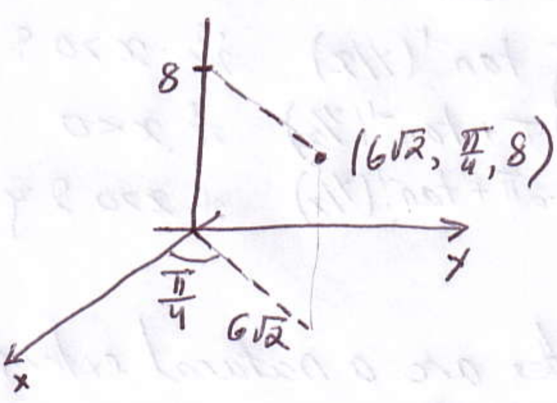


Ex. (a) $(6,6,8)$ is in rectangular coordinates. What are the cylindrical coordinates of this point? (b) If a point has cylindrical coordinates $(8, \frac{2\pi}{3}, -3)$, what are its Cartesian coordinates? Plot.

Solution:

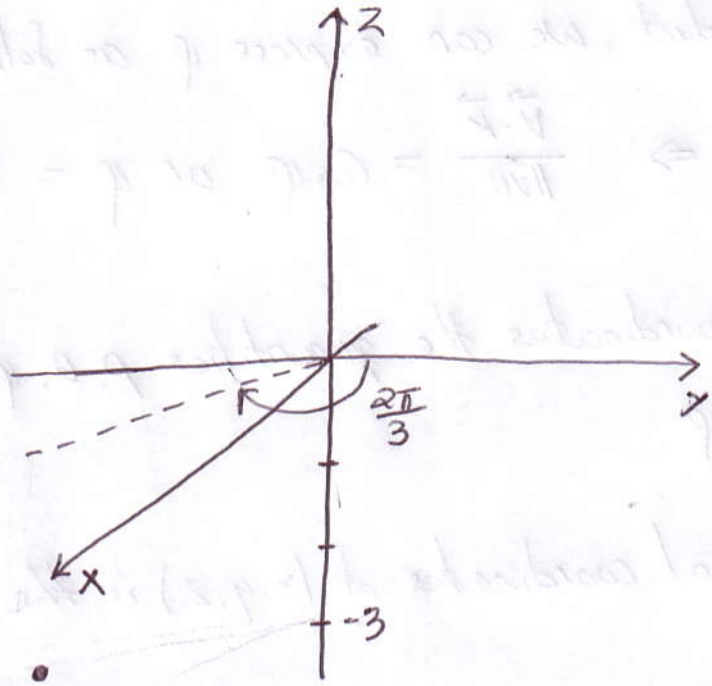
(a) $r=\sqrt{6^2+6^2} = 6\sqrt{2}$, $\theta = \tan^{-1}(6/6) = \tan^{-1}(1) = \frac{\pi}{4}$.

Thus, the cylindrical coordinates are $(6\sqrt{2}, \frac{\pi}{4}, 8)$



(b) $r=8$, $\theta = \frac{2\pi}{3}$, $z=-3$ so $x = r\cos\theta = 8\left(-\frac{1}{2}\right) = -4$

$y = r\sin\theta = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ Hence the cartesian coordinates are $(-4, 4\sqrt{3}, -3)$

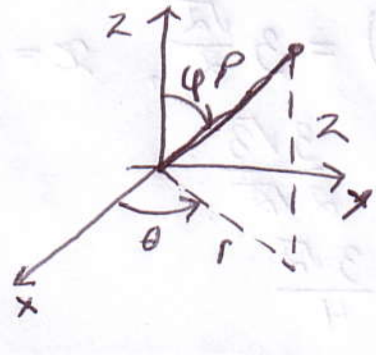


Spherical Coordinates

Given a point $(x, y, z) \in \mathbb{R}^3$, let $\rho = \sqrt{x^2+y^2+z^2}$ and represent x and y by polar coordinates in the xy plane:

$x = r\cos\theta$ $\quad y = r\sin\theta$ where $r = \sqrt{x^2+y^2}$ and θ determined in the same fashion as for cylindrical coordinates.

The coordinate $z = \rho\cos\varphi$ where φ is the angle (chosen to lie between 0 and π inclusive) that the radius vector $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ makes with the positive z axis, in the plane containing the vector $\vec{v}$ and the z-axis.



Using the dot product, we can express φ as follows:

$$\vec{v}\cdot\vec{k} = \|\vec{v}\|\,\|\vec{k}\|\cos\varphi \Rightarrow \frac{\vec{v}\cdot\vec{k}}{\|\vec{v}\|} = \cos\varphi \text{ or } \varphi = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{k}}{\|\vec{v}\|}\right)$$
$$= \cos^{-1}\left(\frac{z}{\rho}\right)$$

We take as our coordinates the quantities ρ, θ, φ. Because
$$r = \rho\sin\varphi$$

Def: The spherical coordinates of (x, y, z) is the triple (ρ, θ, φ) defined as follows:

$$x = \rho\sin\varphi\cos\theta \quad y = \rho\sin\varphi\sin\theta \quad z = \rho\cos\varphi.$$

$$\rho \geq 0 \quad 0 \leq \theta < 2\pi, \quad 0 \leq \varphi \leq \pi.$$

Ex. (a) Find the spherical coordinates of the Cartesian point $(1, -1, 1)$ and plot.

Solution: $\rho = \sqrt{1^2 + (-1)^2 + (1)^2} = \sqrt{3}$, $\theta \in IV$ $\theta = \tan^{-1}\left(\frac{-1}{1}\right) \cancel{2\pi}$

$= 2\pi + \frac{\pi}{4} = \frac{7\pi}{4}$ $\quad \varphi = \cos^{-1}\left(\frac{1}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \approx 54.74°$

(b) Find the Cartesian coordinates of the spherical coordinate point $(3, \frac{\pi}{6}, \frac{\pi}{4})$

Solution: $z = \rho\cos\varphi = 3\cos\left(\frac{\pi}{4}\right) = 3\frac{\sqrt{2}}{2}$, $x = r\cos\theta =$

$$= \frac{3\sqrt{2}}{2}\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2\sqrt{2}}.$$

$$y = r\sin\theta = 3\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{6}\right) = \frac{3\sqrt{2}}{4}$$

Ex. Express in spherical coordinates

(a) the surface $xz=1$

Solution: $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$

Hence $x\cdot z=\rho\sin\varphi\cos\theta\,\rho\cos\varphi=\rho^2\sin\varphi\cos\varphi\cos\theta=1$

or $\frac{1}{2}\rho^2\sin 2\varphi\cos\theta=1 \Rightarrow \rho^2\sin 2\varphi\cos\theta=2$

(b) the surface $x^2+y^2-z^2=1$

Solution:

$$x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2=\rho^2\sin^2\varphi$$

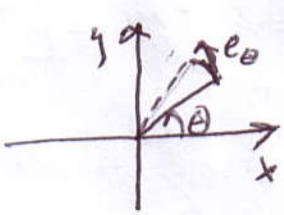
$z^2=\rho^2\cos^2\varphi$

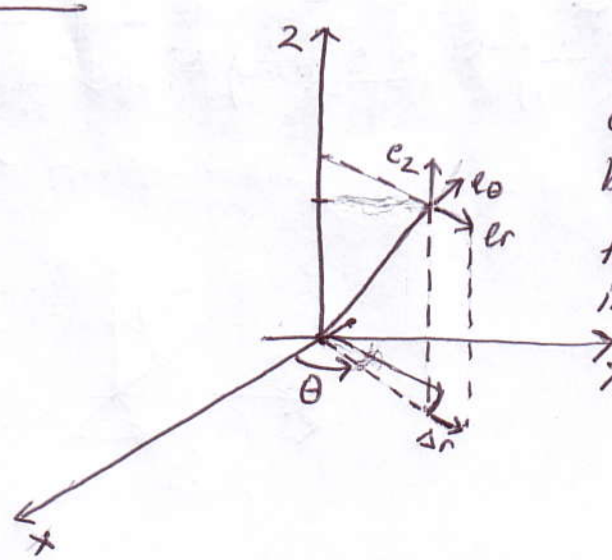
Hence $x^2+y^2-z^2=\rho^2\sin^2\varphi-\rho^2\cos^2\varphi=\rho^2(\sin^2\varphi-\cos^2\varphi)=$

$=-\rho^2(\cos^2\varphi-\sin^2\varphi)=-\rho^2\cos 2\varphi=1.$

Counterparts of $\vec{i},\vec{j},\vec{k}$ in cylindrical and spherical coordinates

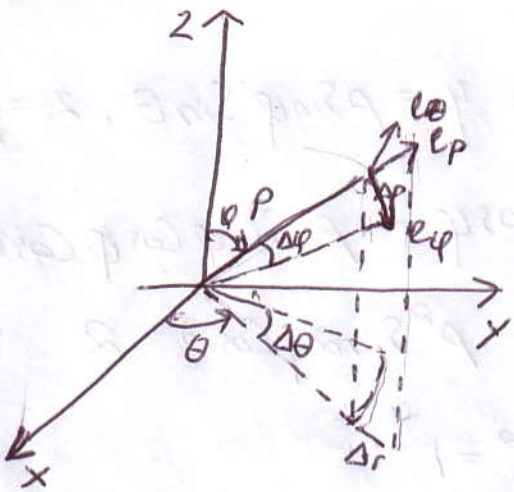
Cylindrical Coordinates





The unit directional vectors for cylindrical coordinates are obtained by connecting a point to its displacement in one of the coordinates

Spherical Coordinates



Notice that e_θ is always parallel to the xy-plane.